Journal of Statistical Physics, Vol. 89, Nos. 5/6, 1997

Super-Instantons, Perfect Actions, Finite-Size Scaling, and the Continuum Limit

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Received April 25, 1996; final April 10, 1997

We discuss some aspects of the continuum limit of some lattice models, in particular the 2D $O(N)$ models. The continuum limit is taken either in an infinite-volume or in a box whose size is a fixed fraction of the infinite-volume correlation length. We point out that in this limit the fluctuations of the lattice variables must be $O(1)$ and thus restore the symmetry which may have been broken by the boundary conditions (b.c.). This is true in particular for the so-called super-instanton b.c. introduced earlier by us. This observation leads to a criterion to assess how close a certain lattice simulation is to the continuum limit and can be applied to uncover the true lattice artefacts, present even in the so-called "perfect actions". It also shows that David's recent claim that super-instanton b.c. require a different renormalization must either be incorrect or an artefact of perturbation theory.

KEY WORDS: Classical spin models; continuum limit.

1. INTRODUCTION

Lattice field theory can be considered as quantum field theory with a cutoff. Of course the challenge is to dispose of the cutoff. From the point of view of the lattice model that means letting the correlation length become large. Combining this requirement with the desire of working in a large thermodynamic box, one is quickly facing forbidding costs in CPU time and memory. Although many techniques have been proposed to circumvent this limitation, generally speaking they fall into the following two categories:

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0022-4715/97/1200-0947$12.50/0

1. Choice of a better lattice action
2. Finite size scaling

Using the first technique one hopes that by complicating sufficiently the lattice action the cutoff effects can be reduced so that continuum behavior can be observed already at a correlation length of a few lattice units with the perfect action of Hasenfratz and Niedermayer.(1)

With the second technique one simulates the system in a box of finite "physical size," i.e. a box whose linear extent L is a certain fraction of the thermodynamic (= infinite volume) correlation length ξ. The idea is then to approach the continuum limit by considering a sequence of lattices with fixed ratio $z = L/\xi$ and extrapolating to the limit $\xi \to \infty$, using certain assumptions about the asymptotic behavior. Thereby it is believed that the so-called lattice artefacts can be eliminated. A notable example of this philosophy is the work of the "Alpha Collaboration" and its precursors.(2) This procedure raises some questions, since the proposed form of the approach to the limit does not have a solid theoretical basis and different assumptions about it lead to quite different estimated values of this limit.(3)

The crucial question is: do these techniques manage to reflect the true continuum behavior up to some small corrections, or are they dominated by lattice artefacts? One way to assess this lies (maybe surprisingly) in studying the dependence of the data upon the boundary conditions (b.c.) and possibly other constraints on individual spins.

In ref. [4] we introduced "super-instanton b.c." (s.i.b.c.) that are characterized by fixing the spins at the boundary and in addition a spin in the middle of the lattice. We pointed out that in the thermodynamic limit one has to obtain the same results with s.i.b.c. as with more conventional b.c.. In this paper we will show that the same is true in the continuum limit even when it is taken in a box of "finite physical size". The practical use of this observation lies in the fact that one can check to what extent this independence is fulfilled for particular coupling parameters and box sizes. Although we do not present any numerical data in this paper, our conclusion is that recent claims which appeared in the literature regarding the continuum limit of $2D$ $O(N)$ models and $4D$ gauge theories are unjustified in that they would not pass this test; likewise is the claim that by employing perfect actions one can observe continuum behavior already at small correlation length.

Our observation that super-instanton b.c. must lead to the same continuum limit as say Dirichlet b.c. answers also David's(5) recent claim that they require a different renormalization: they cannot possibly do so. Consequently, if in perturbation theory (PT) one finds, as he claims, that in fact one does need additional renormalizations with s.i.b.c., then that is another proof that in these models PT fails to produce the correct asymptotic expansion.

Before starting our discussion, we remind the reader of the general procedure used to obtain a continuum limit of a lattice model: first one has to find a point in parameter space where at least one dynamically generated correlation length, called ξ, diverges. Then the continuum correlation functions can be obtained by driving the system into this critical point, using ξ as the standard of length; calling the lattice fields $s(x)$, this means that the n-point continuum correlation function (Schwinger function) is given by

$$S_n(x_1,..., x_n) = \lim_{\xi \to \infty} Z(\xi)^{-n/2} \langle s(x_1 \xi),..., s(x_n \xi) \rangle \quad (1)$$

where $Z(\xi)$ will be a suitably chosen field strength renormalization constant. This will produce a massive continuum limit (of mass 1 with the choice made in Eq. (1)). Alternatively one can construct a massless continuum limit by sitting right at a critical point, introducing an arbitrary length standard L_o that is sent to ∞ and defining

$$S_n(x_1,..., x_n) = \lim_{L_o \to \infty} Z(L_o)^{-n/2} \langle s(x_1 L_o)...s(x_n L_o) \rangle \quad (2)$$

At least if the lattice fields $s(x)$ are bounded, it is unavoidable that the field strength renormalizations $Z(\xi)^{-1/2}$ diverge for $\xi \to \infty$, if the continuum limit is to be a quantum field theory which has by necessity short distance singularities in its Schwinger functions.

2. GAUSSIAN COMPUTATIONS

To get a feeling for the situation, it is useful first to consider free scalar fields Φ on the lattice $\mathbb{Z}^D$ with mass m (including the case $m = 0$). First we look at the continuum limit on an infinite lattice: the field is described by a Gaussian measure with covariance

$$C(x - y) = (-\Delta + m^2)^{-1} (x, y) \quad (3)$$

where Δ is the lattice Laplacian. To obtain the continuum limit, we have to drive the system into the critical point $m = 0$, using $\xi = 1/m$ as the standard of length. In other words, we study correlation functions at distances that are fixed fractions of the correlation length $\xi = 1/m$, e.g.

$$\left\langle \Phi\left(\frac{x}{m}\right) \Phi\left(\frac{y}{m}\right)\right\rangle = C\left(\frac{x-y}{m}\right) \tag{4}$$

and send $m \to 0$.

In dimension $D \geqslant 2$ this 2-point function is $O(m^{D-2})$, so to get a nontrivial continuum limit in $D > 2$ one has to introduce a divergent field strength renormalization $Z(m)$ and define

$$\Phi_r(x) = \Phi\left(\frac{x}{m}\right) Z(m)^{-1/2} \tag{5}$$

with $Z(m) = O(m^{D-2})$. Then one obtains the continuum limit (for $x \neq y$)

$$\lim_{m \to 0} \langle \Phi_r(x)\, \Phi_r(y)\rangle = \frac{1}{(2\pi)^D} \int d^D p \frac{e^{ip\cdot(x-y)}}{p^2+1} \tag{6}$$

(The integral does not exist in the classical sense, but has to be interpreted as follows: assuming without loss of generality that $x_o - y_o \neq 0$, one integrates first over p_o using the calculus of residues; the remaining integral is then absolutely convergent. To show convergence of the renormalized lattice two-point function to the continuum limit, one uses the same trick. After the first integration has been carried out, the dominated convergence theorem can be used.)

In $D = 2$ no field strength renormalization is necessary. But in all $D \geqslant 2$ we find for $x \neq y$

$$\lim_{m \to 0} \langle (\Phi_r(x) - \Phi_r(y))^2 \rangle = \infty, \tag{7}$$

which shows that the fluctuations of the renormalized fields diverge. In $D \geqslant 3$ this divergence is due to the field strength renormalization, whereas in $D = 2$ it is due to the fact that $\lim_{m \to 0} C(0) = \infty$ because of the logarithmic infrared (IR) divergence.

It is easy to convince oneself that the fields $\Phi(x/m)$ and $\Phi(y/m)$ become statistically independent in the continuum limit for $x \neq y$: this is true for any b.c. and any $D \geqslant 2$ and is due to the fact that $C((x-y)/m)))/C(0)$ goes to zero.

Slightly less trivial is the case of an exponential of a free field

$$\Psi(x) = e^{iq\Phi(x)} \tag{8}$$

We limit ourselves to the study of $D = 2$, because that is the most interesting case, and for $D > 2$ the continuum fields would become non-tempered, i.e. the correlation functions would develop exponential singularities. Even though the fields Φ did not require renormalization in $2D$, their exponentials do, as can be found essentially already in Coleman's paper.[7] If we define

$$\Psi_r(x) = Z(m)^{-1/2}\, \Psi(x/m) \tag{9}$$

with $Z(m) = m^{q^2/2\pi}$, the correlation functions of the renormalized fields Ψ_r will have nontrivial continuum limits. But it is interesting to look at the continuum limit from the point of view of the lattice fields $\Psi(x/m)$: then we find restoration of the $O(2)$ symmetry in accordance with the Mermin–Wagner theorem[9] in the limit $m \to 0$. Explicitly

$$\left\langle \Psi\left(\frac{x}{m}\right) \Psi\left(\frac{y}{m}\right)^* \right\rangle = \exp\left(-q^2\left(C(0) - C\left(\frac{x-y}{m}\right)\right)\right) \tag{10}$$

which goes to 0 as $m \to 0$ because $C(0) = O(|\ln(m)|)$.

One can also establish that the fields $\Psi(x/m)$, $\Psi(y/m)$ become statistically independent in the limit $m \to 0$ for $x \neq y$. Furthermore each field $\Psi(x/m)$ will be distributed uniformly on the unit circle in this limit. To see this, it suffices to consider

$$\langle \Psi(x/m)^{n_x}\, \Psi(y/m)^{n_y} \rangle = e^{-(q^2)/2\,[C(0)(n_x^2 + n_y^2) + 2n_x n_y C((x-y)/m))} \tag{11}$$

for $n_x, n_y \in \mathbb{Z}$. It is easy to see that for $n_x^2 + n_y^2 \neq 0$ this goes to zero, while for $n_x = 0 = n_y$ it is equal to 1. Thus we have

$$\lim_{m \to 0} \langle \Psi(x/m)^{n_x}\, \Psi(y/m)^{n_y} \rangle = \delta_{n_x 0} \delta_{n_y 0} \tag{12}$$

from which the claim follows.

Next we turn to the continuum limit in a box. The linear extent L is to be kept fixed in "physical units," i.e. we choose $L = l/m$ with l fixed. We may use Dirichlet, periodic or any other classical b.c.. The discussion of the continuum limit proceeds as above and the equations are changed only by replacing $C(x-y)$ with $C_{bc}(x, y)$, the covariance with the appropriate b.c.; we find as above:

(1) In $D > 2$ the Gaussian field Φ requires a wave function renormalization, leading to divergent fluctuations of the massive free field in the continuum limit.

(2) In $D=2$, Φ requires no wave function renormalization, but the fluctuations of the free massive field diverge in the continuum limit due to the IR divergence of $C_{bc}(x, x)$.

(3) In $D=2$ the renormalized exponentials $\Psi_r(x)$ of the free field require a wave function renormalization; their expectation values are not $O(2)$ symmetric in accordance with the symmetry breaking mass term.

(4) In $D=2$ the unrenormalized exponentials $\Psi(x)$ show restoration of the $O(2)$ symmetry, because from the point of view of the lattice (i.e. measured in lattice units) the box is becoming infinitely large and the symmetry breaking by the mass term disappears in the limit. The fields $\Psi(x_i/m)$ become statistically independent for different x_i.

Next we turn to s.i.b.c.. They are defined as 0 Dirichlet b.c. at the boundary of the box, at distance $O(1/m)$ from the origin, together with the constraint $\Phi(x_c)=0$ where x_c is a point in the "middle of the lattice," e.g. the origin. The Green's function $\langle \Phi(x)\, \Phi(y)\rangle$ with these b.c. can be expressed in terms of the Dirichlet Green's function C_D as follows:[4,8]

$$\langle \Phi(x)\, \Phi(y)\rangle_{s.i.b.c.} = C_D(x, y) - \frac{C_D(x, x_c)\, C_D(x_c, y)}{C_D(x_c, x_c)} \tag{13}$$

More generally we may require that $\Phi(x_c)=a$, whereas at the edges of the box we still have 0 Dirichlet b.c.. In this case the two-point function becomes

$$\langle \Phi(x)\Phi(y)\rangle_{s.i.b.c.} = C_D(x, y) - \frac{C_D(x, x_c)\, C_D(x_c, y)}{C_D(x_c, x_c)} + a^2 \frac{C_D(x, x_c)\, C_D(x_c, y)}{C_D(x_c, x_c)^2} \tag{14}$$

and there is also a nonvanishing one-point function

$$\langle \Phi(x)\rangle_{s.i.b.c.} = a\, \frac{C_D(x_c, x)}{C_D(x_c, x_c)} \tag{15}$$

Looking at Eqs. (13–15) one sees at once that in $2D$, if we replace x by x/m etc., the extra terms go to zero because their denominators blow up as $m \to 0$. In $D>2$, after field strength renormalization, and replacing x by x/m etc, the extra terms go to zero because the numerators do not have enough renormalization factors. It should not come as a surprise that the additional constraint $\Phi(x_c)=a$ does not leave any trace in the continuum limit of the renormalized fields, because in the continuum it is impossible to impose a Dirichlet condition at a point in $D>1$ (or more generally on a set of zero capacity).

It is clear from Eqs. (8) and (9) that the situtation is analogous for the exponential fields $\Psi(x/m)$ and $\Psi_r(x)$.

There is another continuum limit that can be discussed: we can put the massless Gaussian field in a box of size L, with 0 Dirichlet b.c. to avoid trouble from the zero mode, and take L as the standard of length. The continuum limit is now defined by the limit $L \to \infty$ of the correlations of the renormalized fields

$$\Phi_r(x) = Z(L)^{-1/2}\, \Phi(xL) \tag{16}$$

with $Z(L) = L^{2-D}$ $(D \geqslant 2)$, or (only in $2D$):

$$\Psi_r(x) = Z(L)^{-1/2}\, \Psi(xL) \equiv Z(L)^{-1/2}\, e^{iq\Phi(xL)} \tag{17}$$

with $Z(L) = L^{-q^2/2\pi}$.

Also in this massless case we can consider s.i.b.c.. The results are analogous to the massive case discussed above:

(1) The renormalized fields Φ_r show divergent fluctuations as in Eq. (5) in the continuum limit.

(2) The renormalized fields Ψ_r $(D=2)$ show no $O(2)$ symmetry.

(3) The lattice fields Ψ show restoration of the $O(2)$ symmetry.

(4) S.i.b.c. become identical to Dirichlet b.c. in the continuum limit.

3. TWO-DIMENSIONAL $O(N)$ MODELS

In this section we want to show that what we found for the Gaussian models also holds more generally, in particular for the $2D$ $O(N)$ models. These models describe configurations of classical spins $\{s(x)\}$, $s(x) \in \mathbb{R}^N$, $s(x) = 1$, $x \in \Lambda$, where Λ is the lattice $\mathbb{Z}^D$ or a finite part of it (like a box of size L). For definiteness we may consider the standard nearest neighbor action (s.n.n.a.)

$$S = \sum_{\langle xy \rangle} s(x) \cdot s(y) \tag{18}$$

(even though that is inessential) and the Gibbs state induced by it via the Boltzmann factor $\exp(-\beta S)$.

First let us discuss the xy-model $(N=2)$. This model has the famous Kosterlitz–Thouless transition from a massive phase at $\beta < \beta_{crt}$ to a massless one at $\beta \geqslant \beta_{crt}$ [10, 11]. A massive continuum limit is constructed by driving $\beta \to \beta_{crt}$ from below, using the correlation length $\xi = 1/m$ as the unit of length, as in the Gaussian models above. For instance the two-point Schwinger function becomes

$$S_2(x, y) = \lim_{\beta \to \beta_{crt} - 0} Z(\beta)^{-1} \langle (s(x\xi) \cdot s(y\xi)) \rangle \tag{19}$$

First let us point out that there has to be a field strength renormalization $Z(\beta)^{-1}$ that diverges for $\beta \to \beta_{crt} - 0$, to compensate for the fact that without it the two-point function would go to zero. This can be seen as follows: let $\Lambda_{r\xi}$ be a box of size $r\xi$ within the infinite lattice $\mathbb{Z}^D$. Then the root mean square (rms) magnetization in that box is given by

$$M_{rms} = \frac{1}{(r\xi)^2} \sqrt{\left\langle \left(\sum_{x \in \Lambda_{r\xi}} s(x) \right)^2 \right\rangle}. \tag{20}$$

In the $O(2)$ model it follows from Ginibre's inequalities [12] that the two-point function is everywhere nonnegative for a large class of b.c. (including periodic and Dirichlet b.c.). The same is then true for the thermodynamic limit obtained using these b.c.. It follows that M_{rms} will be bounded by $\sqrt{(\chi/(r\xi)^2}$ (remember that there is no subtraction of a disconnected contribution because the $O(2)$ symmetry is unbroken). Now it is well known that

$$\frac{\chi}{\xi^2} = O(\xi^{-\eta}) \tag{21}$$

for $\beta \to \beta_{crt} - 0$, and according to the Kosterlitz–Thouless theory[10] $\eta = 1/4$ (reasonably well confirmed by the numerical simulations).[13,14] This implies that the *rms* magnetization over a box of size $r\xi$ goes to 0 in the continuum limit. But the M_{rms}^2 is nothing but a double average of the two-point functions over that box, and under the positivity assumption made above it follows that

$$\lim_{\beta \to \beta_{crt} - 0} \langle (s(x\xi) \cdot s(y\xi)) \rangle = 0 \tag{22}$$

for $x \neq y$. On the way we have learned that the correlation between two spins located at a distance $x\xi$ will go to zero as $\xi \to \infty$, contrary to what one might have guessed naively (but in agreement with the Ornstein–Zernike behavior as discussed below).

It should be pointed out that the existence of a nontrivial continuum limit requires that the same field strength renormalization that is needed for the two-point function also works for the higher n-point functions.

Composite fields (products of fields a the same point) will require field strength renormalization as well (cf. Eq. (11)). Conversely, since the field strength renormalization has to diverge in the continuum limit, all the correlations of the unrenormalized lattice spins at "physical" distances will go to zero in that limit, and those spins will become statistically independent.

Next let us turn to the continuum limit in a box $\Lambda_{r\xi}$ of size $r\xi$. As before we find that the *rms* magnetization is given by

$$M_{rms} = \frac{1}{(r\xi)^2} \sqrt{\left\langle \left(\sum_{x \in \Lambda_{r\xi}} s(x) \right)^2 \right\rangle_{r\xi}} = \frac{\chi_{r\xi}}{\xi^2} \tag{23}$$

Note that we denote by ξ the infinite volume correlation length and by $\chi_{r\xi}$ the susceptibility in the finite box. We invoke now the hypothesis of finite size scaling (FSS),[15] which says that

$$\lim_{\beta \to \beta_{crt} - 0} \frac{\chi_{r\xi}}{\chi_\infty} = f(r) \tag{24}$$

to conclude as before that also the two-point function in the box will go to zero in the limit $\xi \to \infty$.

As in the Gaussian models, one can also discuss a continuum limit in the massless (KT) phase. Since there is no mass to set the scale, one chooses an arbitrary diverging scale unit L_o and considers the limit

$$S_2(x, y) = \lim_{L_o \to \infty} Z(L_o)^{-1} \langle (s(xL_o) \cdot s(yL_o)) \rangle \tag{25}$$

Note that $\beta \geqslant \beta_{crt}$ is kept fixed in this limit. It is known[11, 20] that $\eta > 0$ (KT theory predicts in fact $1/4 \geqslant \eta > 0$), so we can conclude as above that the spin-spin correlation without field strength renormalization will vanish in the continuum limit and a similar argument can be made for the continuum limit in a finite box of sixe L_o.

This conclusion can in fact be drawn already on account of the Mermin–Wagner theorem[9] alone, which is valid for any $\beta < \infty$. This theorem implies the vanishing of the *rms* magnetization in the limit of infinite box size, simply because the two-point function goes to zero as the separation of the points goes to infinity. By this theorem the $O(2)$ symmetry of the unrenormalized spins is restored in the limit, no matter what symmetry breaking b.c. we used.

Now we are ready to discuss s.i.b.c. for the $O(2)$ model. Again they are defined by fixing a spin in the middle in addition to imposing Dirichlet (fixed) b.c. at the boundary of our box. Because the spin in the middle becomes uncorrelated with all the spins that have a distance $O(\xi)$ or

$O(L_o)$, respectively, in both limits (massive and massless) discussed above, s.i.b.c. become equivalent to Dirichlet b.c..

Let us now extend the discussion to the $O(N)$ models with $N>2$. Elsewhere[16,17] we have presented arguments for the existence of a finite β_{crt} such that for $\beta \geqslant \beta_{crt}$ there is a massless phase in all these models. Accepting this point of view, the discussion can be taken over from the $O(2)$ model.

One remark should be made, however, concerning the nonnegativity of the two-point function that is needed for the argument to go through: if we condition on the configuration of the "transverse" components of the spins ($s_\perp = (s_1,..., s_{N-1})$), it follows by Ginibre's inequalities that

$$\langle s_N(0)\, s_N(x)\rangle_{\{s_\perp\}} \geqslant 0 \tag{26}$$

and by averaging over the transverse components with the appropriate measure one sees that $\langle s_N(0)\, s_N(x)\rangle \geqslant 0$ and using $O(N)$ invariance, it follows that the invariant two-point function is nonnegative.

So if $\beta_{crt} < \infty$, the arguments based on the *rms* magnetization for the $O(2)$ model also apply to the $ON)$ models with $N>2$.

The conventional wisdom—with which we disagree—states, however, that the model is critical only at $\beta = \infty$. We want to point out that even if we accept this point of view for the sake of the argument, the same conclusions as before hold.

Since by assumption there is no massless phase, we only have to discuss the massive continuum limit. According to the conventional wisdom the correlation length ξ and the magnetic susceptibility χ behave as follows for $\beta \to \infty$:[18]

$$\xi \propto \beta^{-1/(N-2)} e^{2\pi\beta/(N-2)} \tag{27}$$

$$\chi \propto \beta^{-(N+1)/(N-2)} e^{4\pi\beta/(N-2)} \tag{28}$$

which would imply

$$\frac{\chi}{\xi^2} = O(\beta^{-(N-1)/(N-2)}) \tag{29}$$

Since this vanishes in the limit $\beta \to \infty$, we obtain again the conclusion that in the limit $\xi \to \infty$ the spin-spin correlations at distances that are fixed fractions of the correlation length will vanish and the system, from the point of view of the lattice spins, restores the $O(N)$ symmetry in that limit.

Therefore quite generally, and independently of the question whether the conventional scenario is true or false, the existence of a continuum limit describing a quantum field theory enforces a divergent field strength renormalization, and as above we conclude that therefore the lattice spins at distances proportional to the correlation length will become statistically independent of each other.

Another understanding of this important point is provided by PT. Indeed tree level PT (which is uncontested by anybody) can be used to estimate the scale over which the spins stay well aligned: in a region where the fluctuations of the spins are small, they are to a good approximation Gaussian, i.e. given by tree level PT. The Gaussian approximation breaks down, however, where it predicts fluctutations that are $O(1)$. This happens at a scale

$$\xi_{PT} = e^{2\pi\beta/(N-1)} \tag{30}$$

so the scale where fluctuations remain small cannot grow faster than this length ξ_{PT}. This length is becoming arbitrarily small with respect to the correlation length ξ, whether we subscribe to the conventional scenario (Eqs. (27, 28)) or believe in the existence of a critical point at finite β. So this consideration also leads to the conclusion reached earlier that the spins decorrelate over distances of the order $O(\xi)$ as $\xi \to \infty$. Conversely, if the distance over which the spins remain well aligned were of the same order as the correlation length ξ then the $O(N)$ symmetry could not possibly be restored at finite fractions of ξ, as it must, according to the arguments presented before.

Likewise it follows that s.i.b.c. are equivalent to Dirichlet b.c. in the continuum. An easy generalization is that one may also fix any finite number of spins provided the distances between them are fixed fractions of the correlation length; in the continuum limit this will have no effect on the correlation functions. This statement has to be interpreted in the sense of distributions, because it is not true for certain exceptional points, for instance if some of the spins whose correlations are considered happen to be the fixed ones. But these exceptional points do not play any role if we smear with test functions. An illustration of these phenomena is easily obtained by Gaussian calculations along the lines of Section 2.

The necessity of a divergent field strength renormalization (and the absence of spontaneous symmetry breaking at any β, which in $2D$ follows from the Mermin-Wagner theorem), is also in full accordance with the so-called Ornstein–Zernike behavior of correlations at large distance (see for instance ref. [19]); one can even obtain a prediction for the behavior of the field strength renormalization from the Ornstein–Zernike behavior. The Ornstein–Zernike postulate, which has been proven in some cases like the Ising model,[19] but is expected to hold generally in massive models, since it corresponds to the requirement that the model describes massive particles with an isolated mass shell, says:

$$\langle s(x\xi)\cdot s(y\xi)\rangle \cong \xi^{2-D-\eta}\,|x-y|^{-(D-1)/2}\exp(-|x-y|) \tag{31}$$

for $|x-y| \gg 1$. It can be seen immediately that this expression times a field strength renormalization factor Z^{-1} has a continuum limit if and only if

$$Z = O(\xi^{2-D-\eta}) \tag{32}$$

The Gaussian models discussed in the previous section have $\eta = 0$; so Eq. (30) generalizes the result found there. From the so-called infrared bounds it follows that $\eta \geqslant 0$ (see for instance ref. [20]), an inequality that is also required if the continuum theory is to be Osterwalder–Schrader positive; of course logarithmic corrections to the pure power behavior assumed in Eq. (29) are legitimate, provided they correspond to a stronger divergence of Z than in the free Gaussian model. In all these cases the Ornstein–Zernike behavior leads to the same conclusions as our other considerations.

4. DISCUSSION AND CONCLUSIONS

We have seen that fixing a finite number of spins has no effect on the continuum limit in a box of finite physical size or in the infinite volume. This is to be contrasted with the claim made by David[5] in a comment to our papers.[4,6] David claimed that fixing a spin at the origin in addition to Dirichlet b.c. (imposing s.i.b.c.) will necessitate extra renormalizations; as we have seen here, there is no effect of the extra spin on the continuum limit. So if in fact in perturbation theory one finds the need for such an extra renormalization, then this is just another proof that perturbation theory does not produce the correct asymptotic expansion in these models.

Furthermore we have learned that in the continuum limit the system gets disordered on the scale of the correlation length, in the sense that spins located at a fixed finite fraction of the correlation length will become decorrelated and this phenomenon even occurs in a box of finite physical size. This is in accordance with the properties of continuum quantum fields known from axiomatic quantum field theory. We have learned there long ago that continuum quantum fields and their correlations are never functions, but have distributional character, in other words there are large fluctuations at short continuum distances. So if the phenomena found here did not occur, there would be no chance to have a continuum limit satisfying e.g. the Osterwalder–Schrader axioms.[21]

In refs. [4, 6] we pointed out the importance of certain "defects" dubbed super-instantons in disordering the $O(N)$ models at large β. These are configurations that turn the spin gradually from a certain value in the center of a region to a different one at its edge; we stressed that these configurations require arbitrarily little energy and have entropy corresponding to their position and scale; therefore they should be abundantly present even at low temperature. Here we found that at the scale of the correlation length, constraining the spin at the center has no effect. This is also a manifestation of the fact that super-instantons become so abundant in the continuum limit that they disorder the system even at the scale of the correlation length, and thus forcing an extra super-instanton into the system has no effect.

There is a useful lesson to be drawn from our observations: since we now know that in the continuum limit fixing a finite number of spins cannot have any effect, by doing precisely this and checking how much the physics changes, we can assess how close our results are to the true continuum limit. This is of relevance in particular in studies where small lattices are used to extract information about the presumed continuum limit; notable examples are the work of the "Alpha collaboration"[2] studying the running coupling, the work by Kim[22] and by Caracciolo *et al.*[23] on finite size scaling in the $O(3)$ model. Some of the claims made there were studied by as more quantitatively in ref. [3], with the conclusion that typically the lattices used were too small to see the true continuum behavior. Our findings in the present paper are also relevant for various claims which have appeared in the literature regarding the miraculous properties of the improved/perfect actions in simulating continuum physics already at rather small correlation lenghts.[1, 24] We would like to state here that this type of claims, that on lattices of modest size the lattice fields can show true continuum behavior, are in our opinion false: whatever the action may be, the lattice must be large enough to allow the typical configuration to resemble a gas of super-instantons, i.e. to restore a certain symmetry required by the Gibbs measure.

Given a lattice model, there exists in principle the possibility to construct a perfect (lattice) action for it: one constructs first the continuum limit, then blocks the continuum theory, thus deriving a perfect action (describing the interaction of the block spin variables) and perfect (continuum) variables (defined in terms of the block variables). From a practical point of view, this procedure is clearly useless since its implementation requires prior control of the contiuum limit.

The necessity of using sufficiently large lattices applies equally to gauge theories. Contrary to the claim of the "Alpha Collaboration" and its precursors, we believe that their data on the running of $\alpha_s(Q)$ do not reveal the true continuum behavior of QCD. Indeed for example the study of this running in $SU(2)^{(2)}$ involves lattices with $L \leqslant 20$ and $\beta \leqslant 3$, a regime in which the typical configuration corresponds to small fluctuations around a well ordered state, rather than a gas of super-instantons; this explains also the excellent agreement they found in the running of $\alpha_s(Q)$ with the prediction of perturbation theory.

ACKNOWLEDGMENTS

We are grateful to the referees for reminding us of the usefulness of Ginibre's inequalities in proving positivity of correlation functions in the $O(N)$ models. A.P. is grateful to the Alexander-von-Humboldt foundation for its support.

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